Indirect Proofs

Announcements

- ***Pset 0***
  - If you missed the deadline, you may still submit.
- ***Pset 1***
  - Goes out today, due next Friday
  - You'll want to watch the video of the LaTeX Beginner's Quick Start Tutorial on Canvas (LaTeX is the preferred tool for writing homework in this class)
  - Partners are allowed—go to Ed Q&A forum to find one
- ***Office Hours***
  - They start Monday! Schedule will be on Canvas later today.

Outline for Today

- ***What is an Implication?***
  - Understanding a key type of mathematical statement.
- ***Negations and their Applications***
  - How do you show something is *not* true?
- ***Proof by Contrapositive***
  - What's a contrapositive?
  - And some applications!
- ***Proof by Contradiction***
  - The basic method.
  - And some applications!

Logical Implication

If n is an even integer, then n^2 is an even integer.

An ***implication*** is a statement of the form "If P is true, then Q is true."

If n is an even integer, then n^2 is an even integer.

This part of the implication is called the ***antecedent***.

This part of the implication is called the ***consequent***.

An ***implication*** is a statement of the form "If P is true, then Q is true."

If n is an even integer, then n^2 is an even integer.

If m and n are odd integers, then $m+n$ is even.

If you like the way you look that much,
then you should go and love yourself.

An ***implication*** is a statement of the form
"If P is true, then Q is true."

What Implications Mean

"If there's a rainbow in the sky, then it's raining somewhere."

- In mathematics, implication is directional.
  - The above statement doesn't mean that if it's raining somewhere, there has to be a rainbow.
- In mathematics, implications only say something about the consequent when the antecedent is true.
  - If there's no rainbow, it doesn't mean there's no rain.
- In mathematics, implication says nothing about causality.
  - Rainbows do not cause rain. 😃

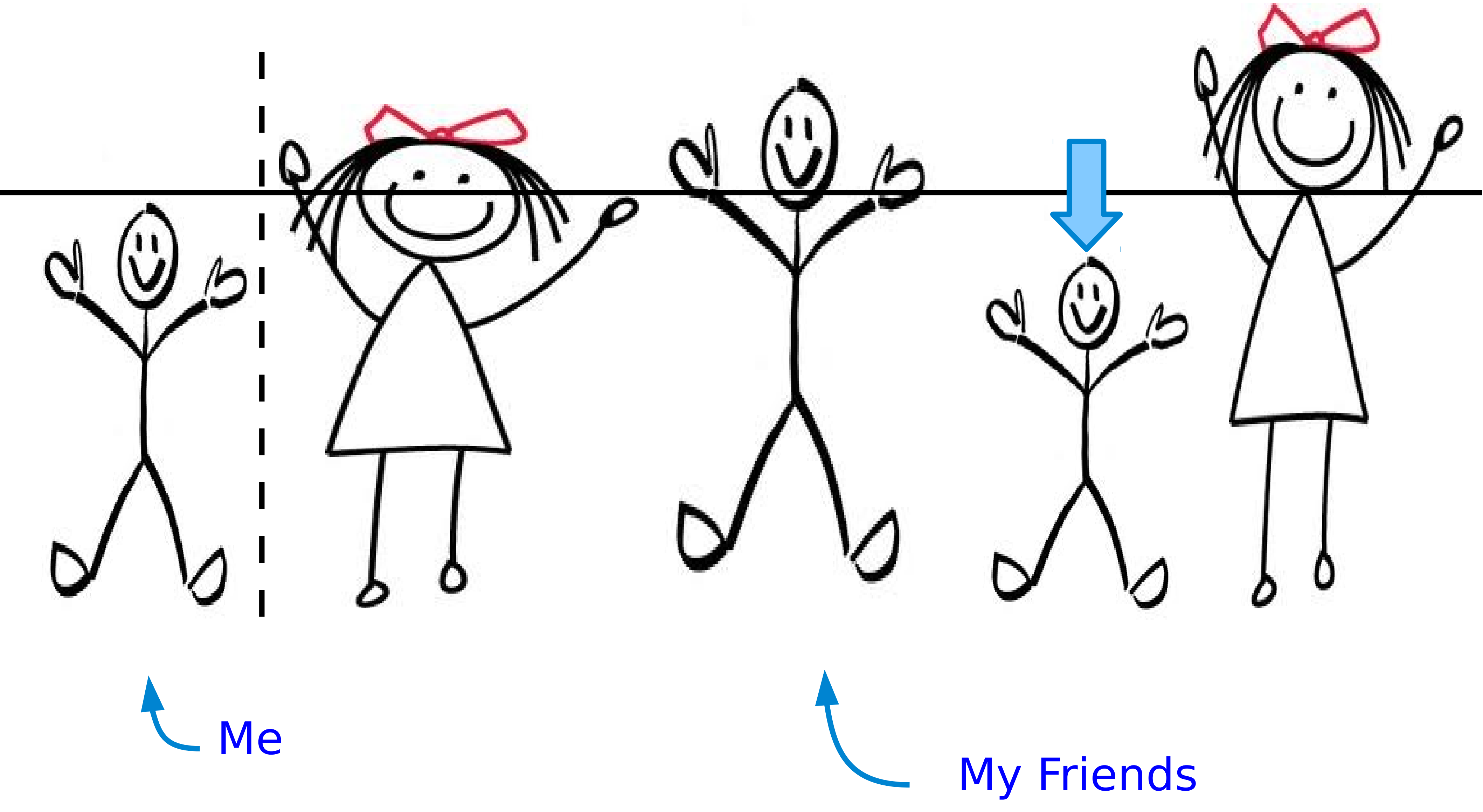
Negations

Negations

- A ***proposition*** is a statement that is either true or false.
- Some examples:
  - If n is an even integer, then n^2 is an even integer.
  - $\emptyset = \mathbb{R}$.
- The ***negation*** of a proposition X is a proposition that is true whenever X is false and is false whenever X is true.
- For example, consider the proposition "it is snowing outside."
  - Its negation is "it is not snowing outside."
  - Its negation is *not* "it is sunny outside." ⚠

How do you find the negation of a statement?

“All My Friends Are Taller Than Me”



The negation of the ***universal*** statement

Every *P* is a *Q*

is the ***existential*** statement

There is a *P* that is not a *Q*.

(Remember that existential means "at least one.")

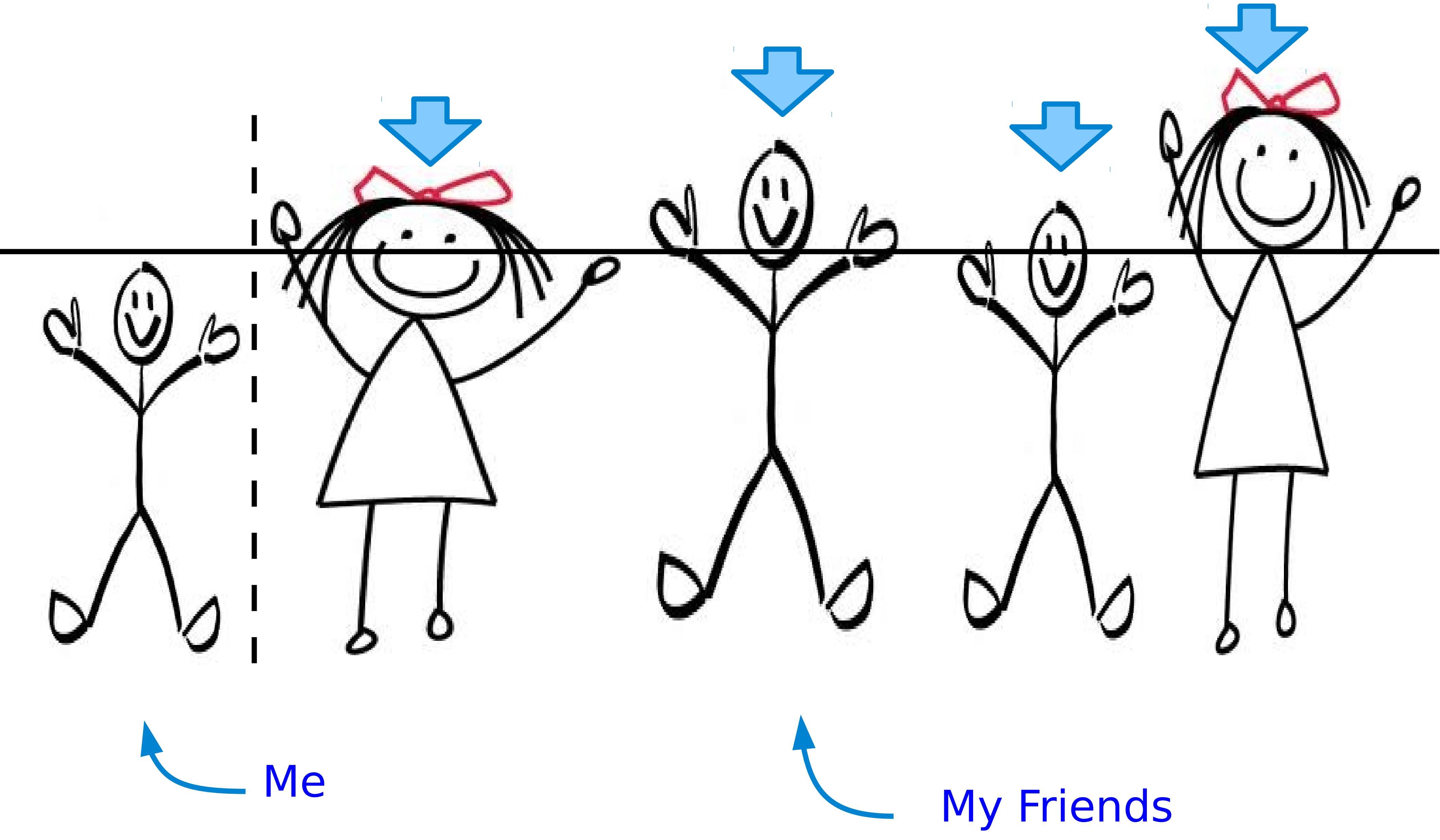
The negation of the ***universal*** statement

For all *x*, *P*(*x*) is true.

is the ***existential*** statement

There exists an *x* where *P*(*x*) is false.

“Some Friend Is Shorter Than Me”



The negation of the ***existential*** statement

There exists a *P* that is a *Q*

is the ***universal*** statement

Every *P* is not a *Q*.

The negation of the ***existential*** statement

There exists an *x* where *P*(*x*) is true

is the ***universal*** statement

For all *x*, *P*(*x*) is false.

How do you negate an implication?

Negating Implication

Dr. Lee: "**If** you pick a perfect March Madness bracket this year, **then** I'll give you an A+ in CS103."

Q: under what conditions am I a liar?*
What if…

- …you pick a perfect bracket and get an A+?
- …you pick a bad bracket and get an A+?
- …you pick a perfect bracket and get a C?
- …you pick a bad bracket and get a C?

** The way we define negation in logic means these are the conditions under which the negation of my statement is true.*

The negation of the statement

“For any *x*, if *P*(*x*) is true, then *Q*(*x*) is true”

is the statement

“There is at least one *x* where *P*(*x*) is true and *Q*(*x*) is false.”

The negation of an implication is not an implication!

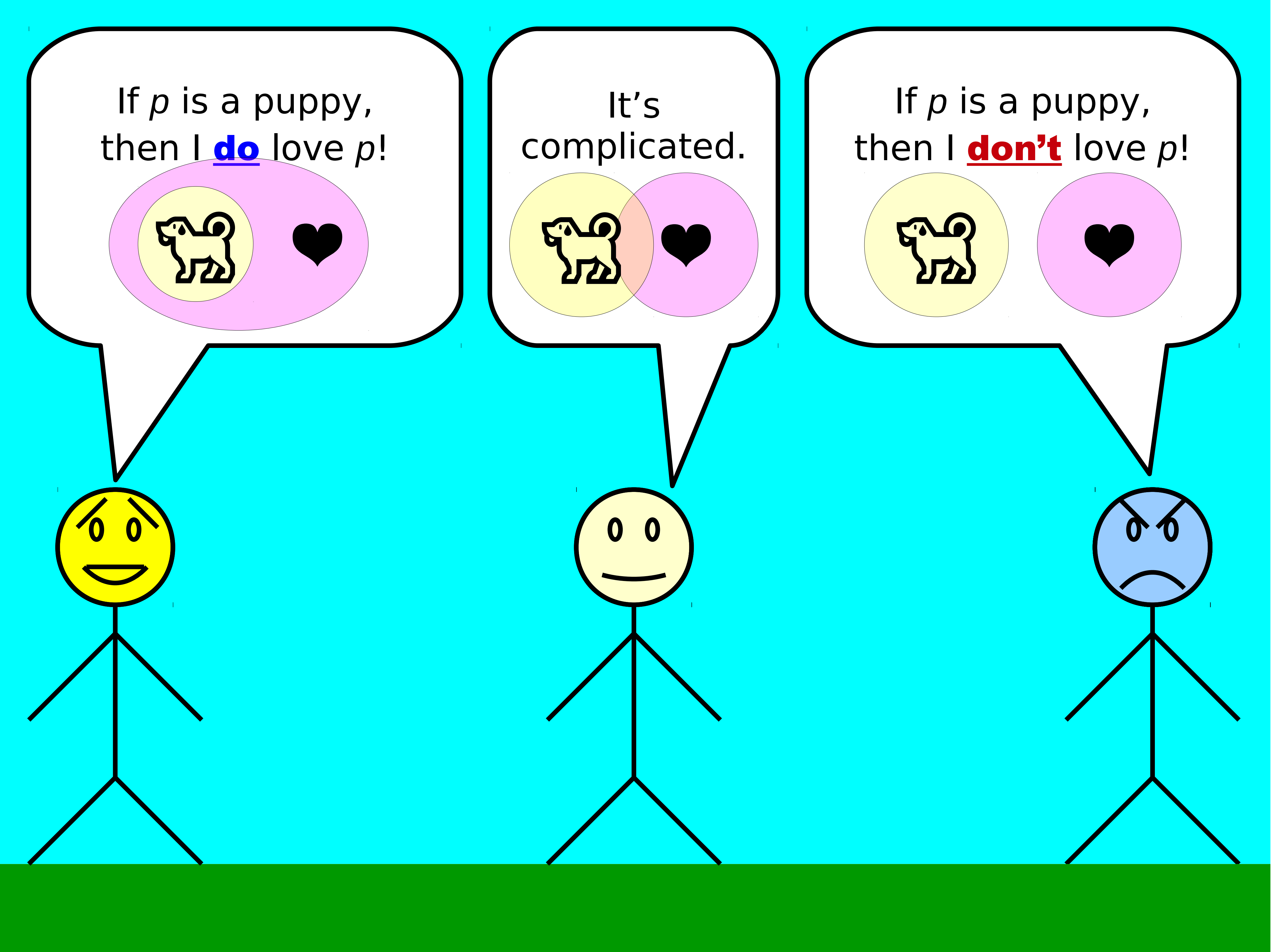
The negation of the statement

"For any x, if $P(x)$ is true, then $Q(x)$ is true"

is the statement

"There is at least one x where $P(x)$ is true and $Q(x)$ is false."

The negation of an implication is not an implication!


If *p* is a puppy, then I **do** love *p*!
It’s complicated.
If *p* is a puppy, then I **don’t** love *p*!

How to Negate Universal Statements:

"For all *x*, *P*(*x*) is true"

becomes

"There is an *x* where *P*(*x*) is false."

How to Negate Existential Statements:

"There exists an *x* where *P*(*x*) is true"

becomes

"For all *x*, *P*(*x*) is false."

How to Negate Implications:

"For every *x*, if *P*(*x*) is true, then *Q*(*x*) is true"

becomes

"There is an *x* where *P*(*x*) is true and *Q*(*x*) is false."

Proof by Contrapositive

If P is true, then Q is true.

If Q is false, then P is false.

What are the negations of the above two statements?

If P is true, then Q is true.

negates to

P is true and Q is false.

If Q is false, then P is false.

What are the negations of the above two statements?

If P is true, then Q is true.

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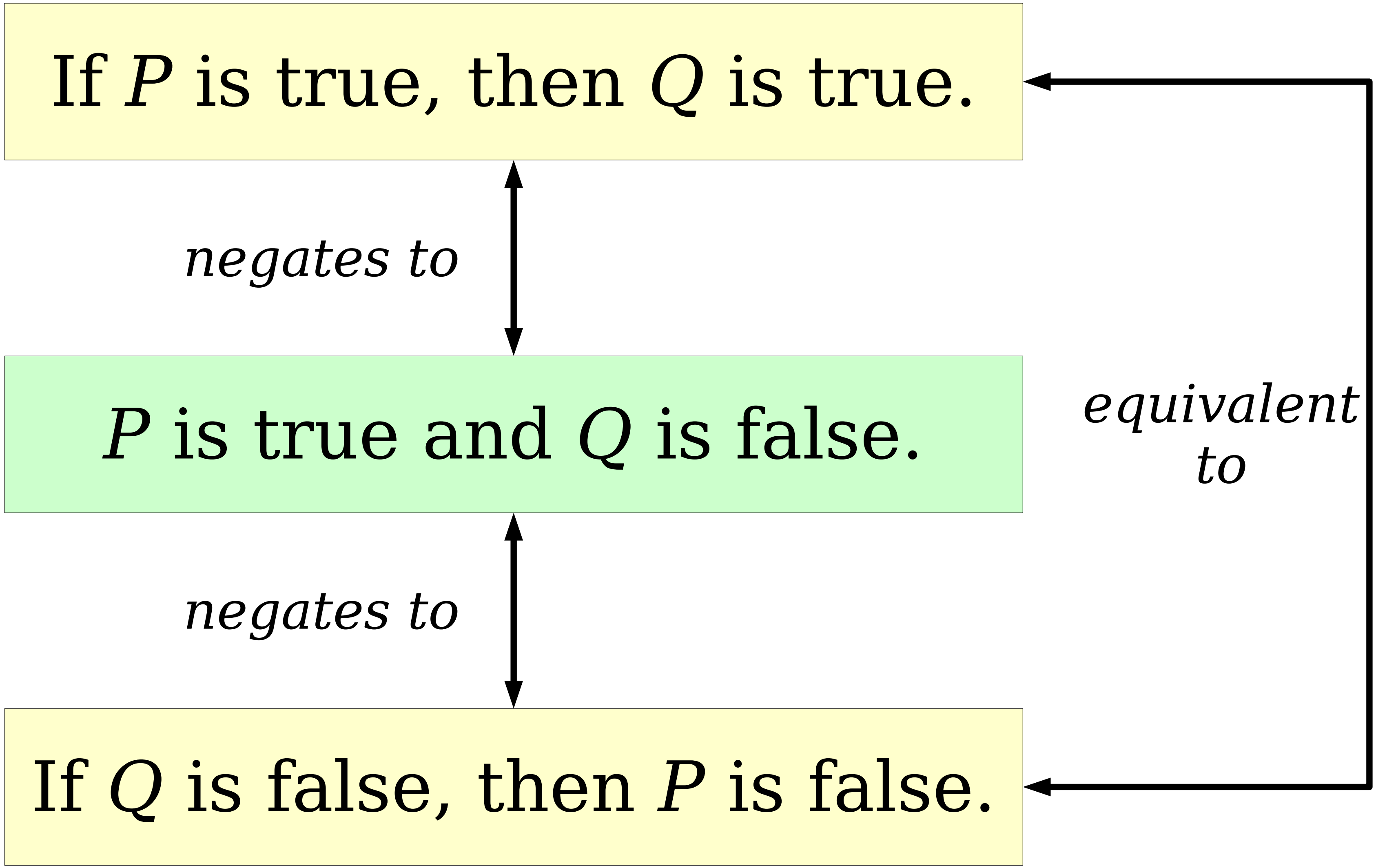

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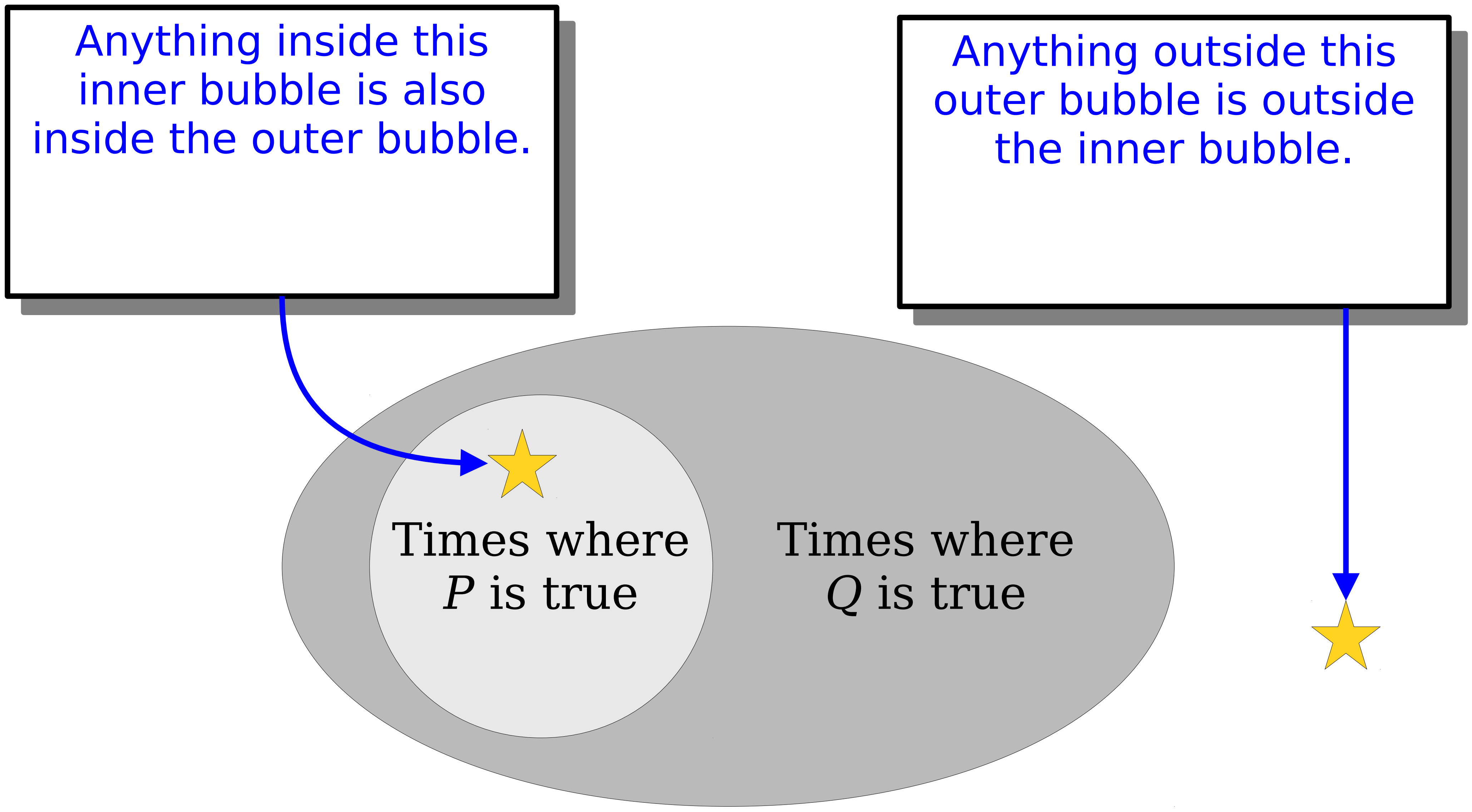


If P is true, then Q is true.

If Q is false, then P is false.

The Contrapositive

- The ***contrapositive*** of the implication

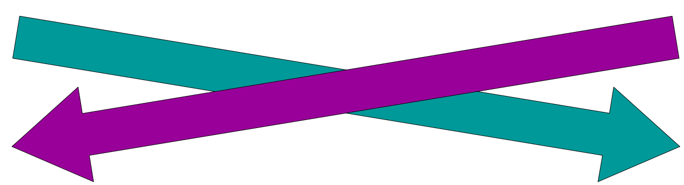
  If *P* is true, then *Q* is true

  is the implication

  If *Q* is false, then *P* is false.

- The contrapositive of an implication means exactly the same thing as the implication itself.

If it's a puppy, then I love it.



If I don't love it, then it's not a puppy.

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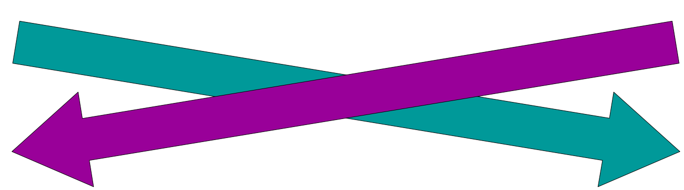
- The contrapositive of an implication means exactly the same thing as the implication itself.

If I store cat food inside, then raccoons won't steal it.



If raccoons stole the cat food, then I didn't store it inside.

To prove the statement

“if *P* is true, then *Q* is true,”

you can choose to instead prove the equivalent statement

“if *Q* is false, then *P* is false,”

if that seems easier.

This is called a ***proof by contrapositive***.

Theorem: For any $n \in \mathbb{Z}$, if n^2 is even, then n is even.

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This is a courtesy to the reader and says "heads up! we're not going to do a regular old-fashioned direct proof here."

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Proof: We will prove the contrapositive of this statement.

What is the contrapositive of this statement?

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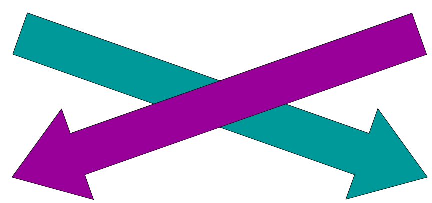

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If n is odd, then n^2 is odd.

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Proof: We will prove the contrapositive of this statement, that if n is odd, then n^2 is odd.

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Theorem: For any $n \in \mathbb{Z}$, if n^2 is even, then n is even.

Proof: We will prove the contrapositive of this statement, that if n is odd, then n^2 is odd.

Here, we're explicitly writing out the contrapositive. This tells the reader what we're going to prove. It also acts as a sanity check by forcing us to write out what we think the contrapositive is.

Theorem: For any $n \in \mathbb{Z}$, if n^2 is even, then n is even.

Proof: We will prove the contrapositive of this statement, that if n is odd, then n^2 is odd.

We've said that we're going to prove this new implication, so let's go do it! The rest of this proof will look a lot like a standard direct proof.

Theorem: For any $n \in \mathbb{Z}$, if n^2 is even, then n is even.

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We know that n is odd, which means there is an integer k such that $n = 2k + 1$.

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$$n^2 = (2k + 1)^2$$

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The general pattern here is the following:

1. Start by announcing that we're going to use a proof by contrapositive so that the reader knows what to expect.

2. Explicitly state the contrapositive of what we want to prove.

3. Go prove the contrapositive.

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Biconditionals

- The previous theorem, combined with what we saw on Wednesday, tells us the following:

  For any integer n, if n is even, then n^2 is even.

  For any integer n, if n^2 is even, then n is even.

- These are two different implications, each going the other way.
- We use the phrase ***if and only if*** to indicate that two statements imply one another.
- For example, we might combine the two above statements to say

  for any integer n: n is even if and only if n^2 is even.

Proving Biconditionals

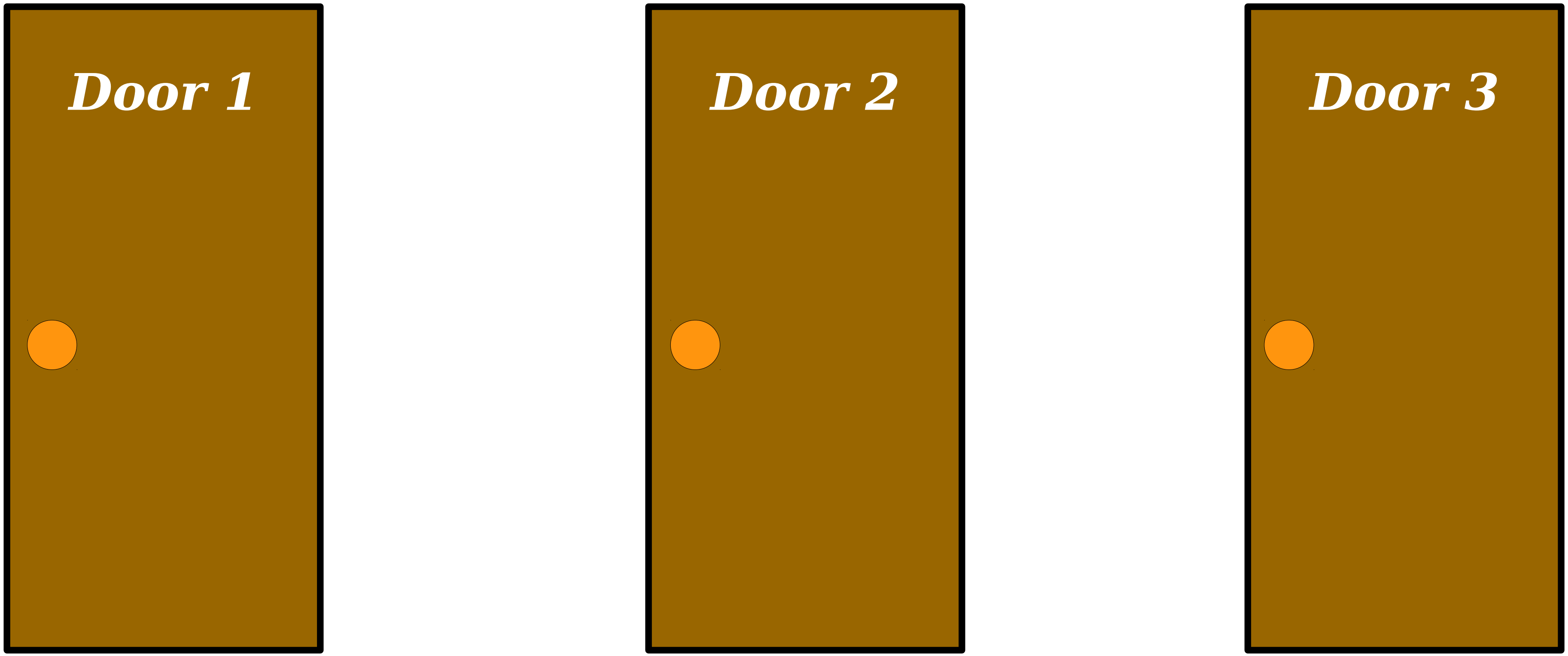
- To prove a theorem of the form

  ***P* if and only if *Q*,**

  you need to prove two separate statements.
  - First, that if P is true, then Q is true.
  - Second, that if Q is true, then P is true.
- You can use any proof techniques you'd like to show each of these statements.
  - In our case, we used a direct proof for one and a proof by contrapositive for the other.

Proof by Contradiction

There's something hidden behind one of these doors. Which door is it hidden behind?



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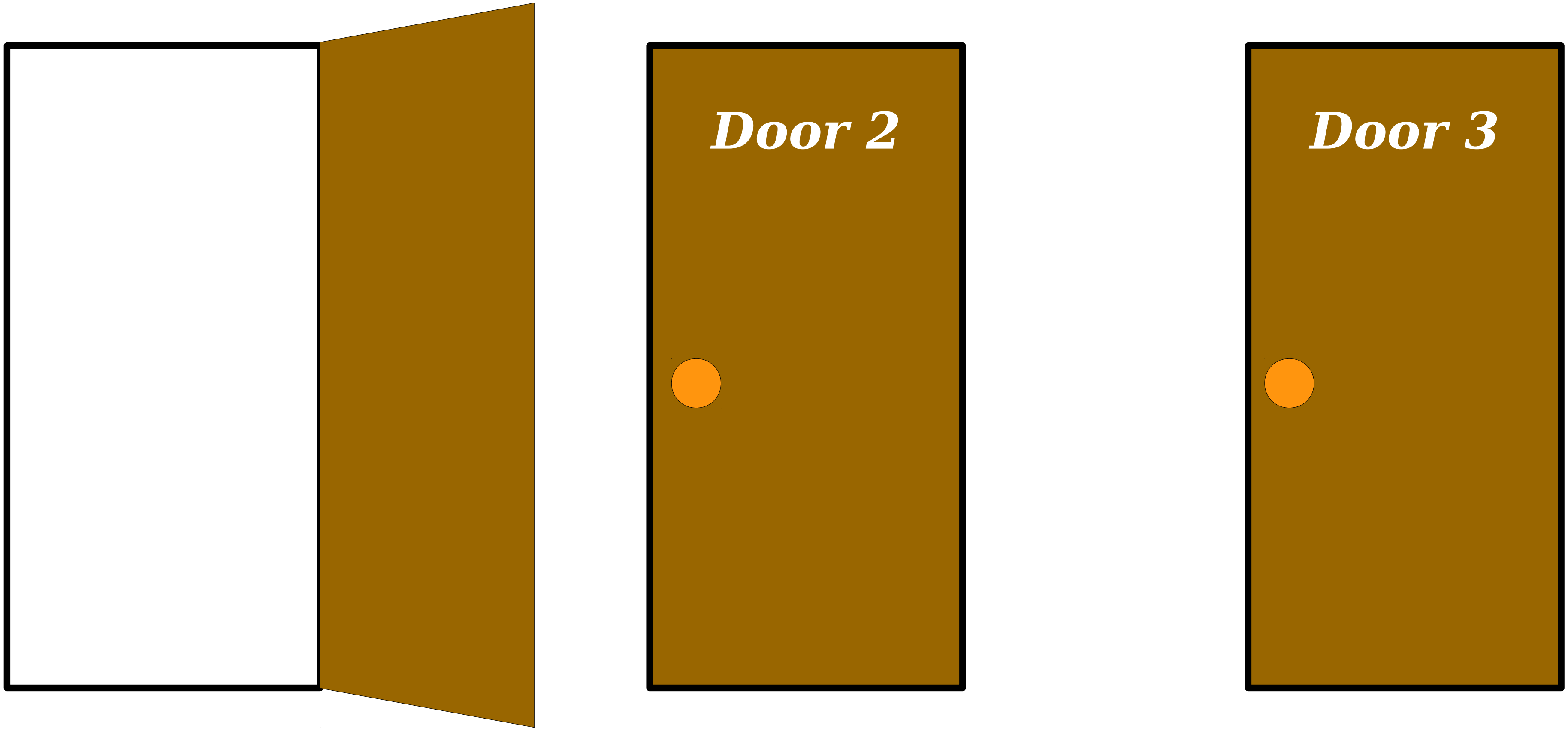

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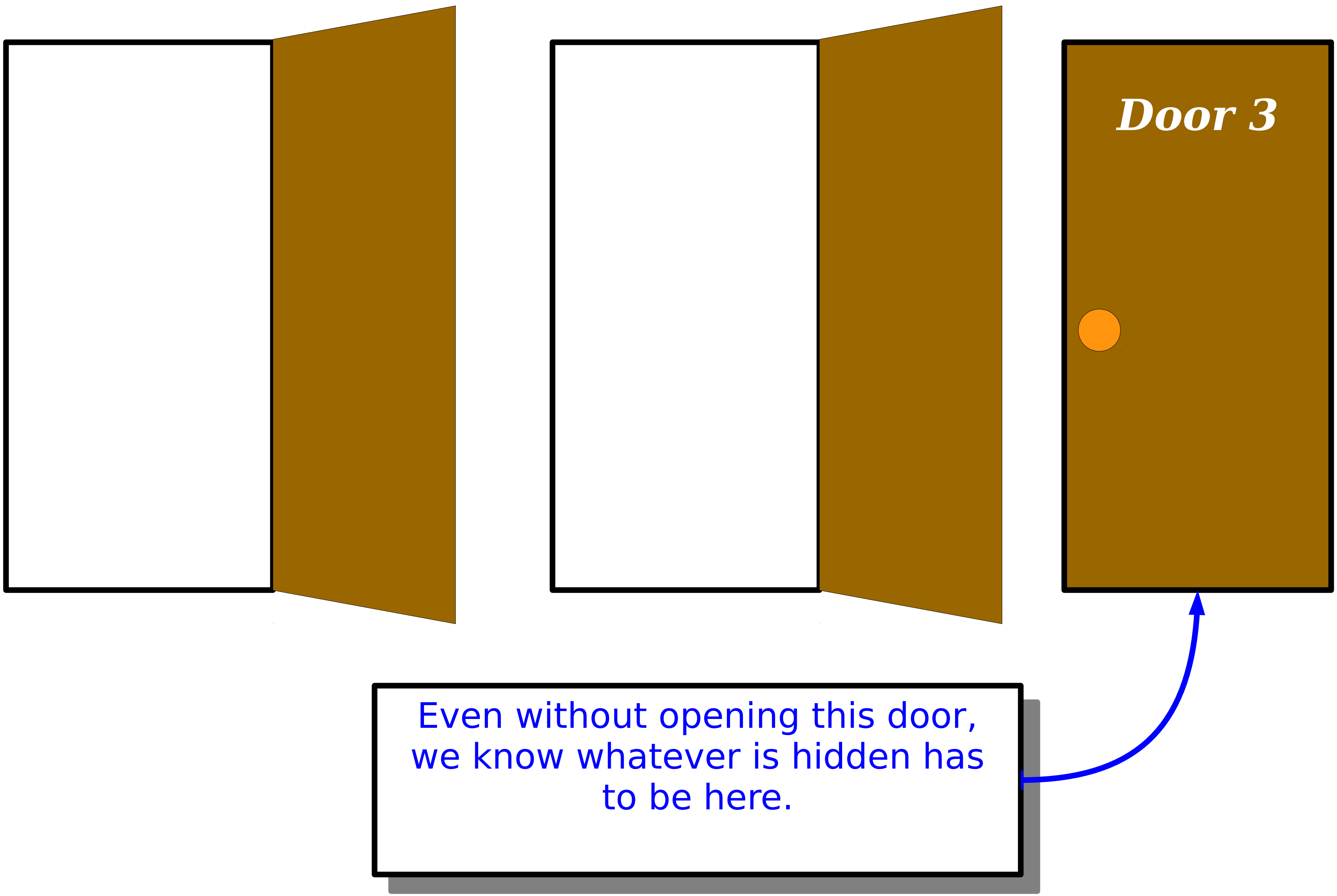

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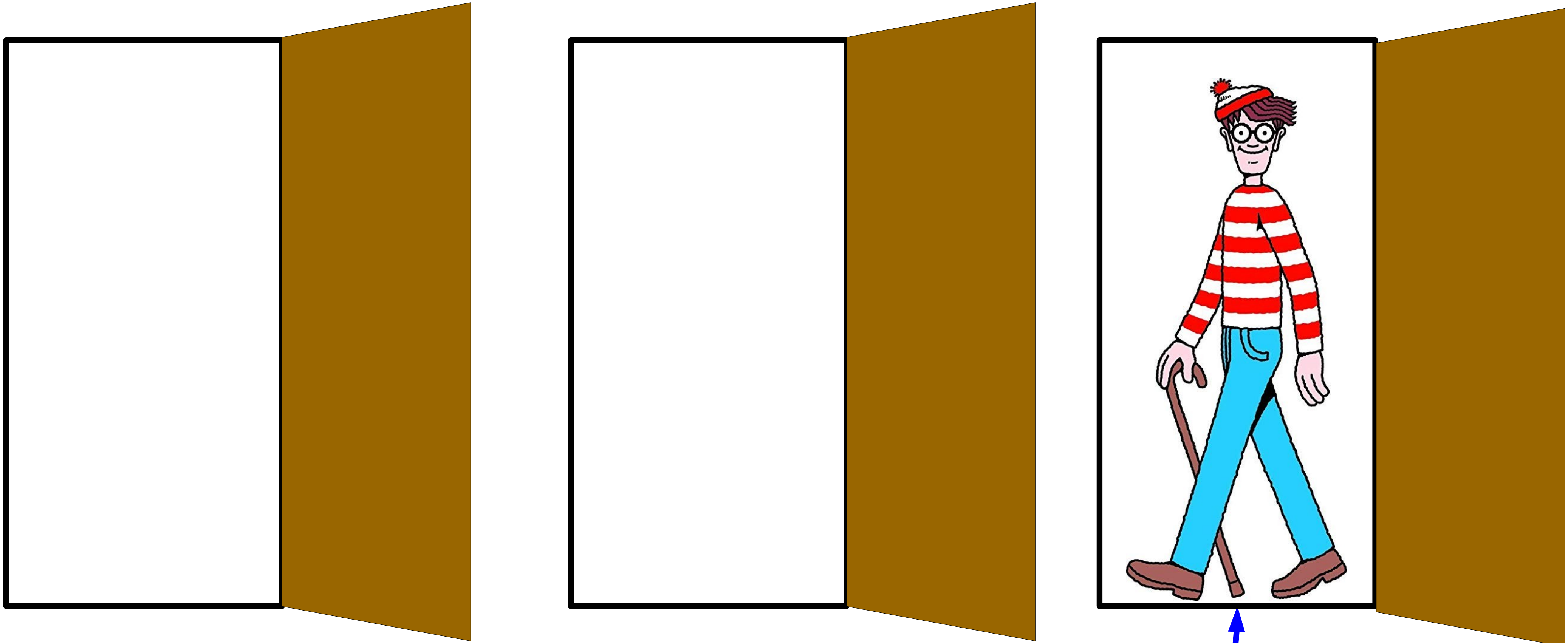
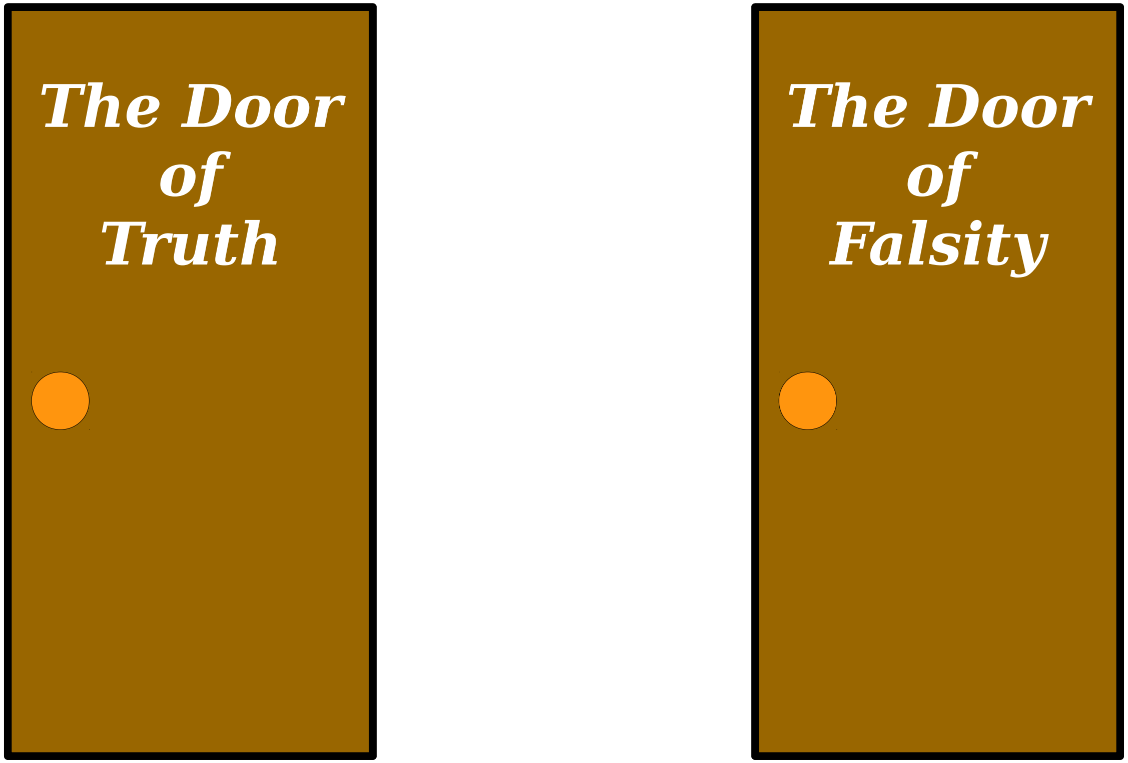


Even without opening this door, we know whatever is hidden has to be here.

Every statement in mathematics is either true or false.
If statement P is not false, what does that tell you?



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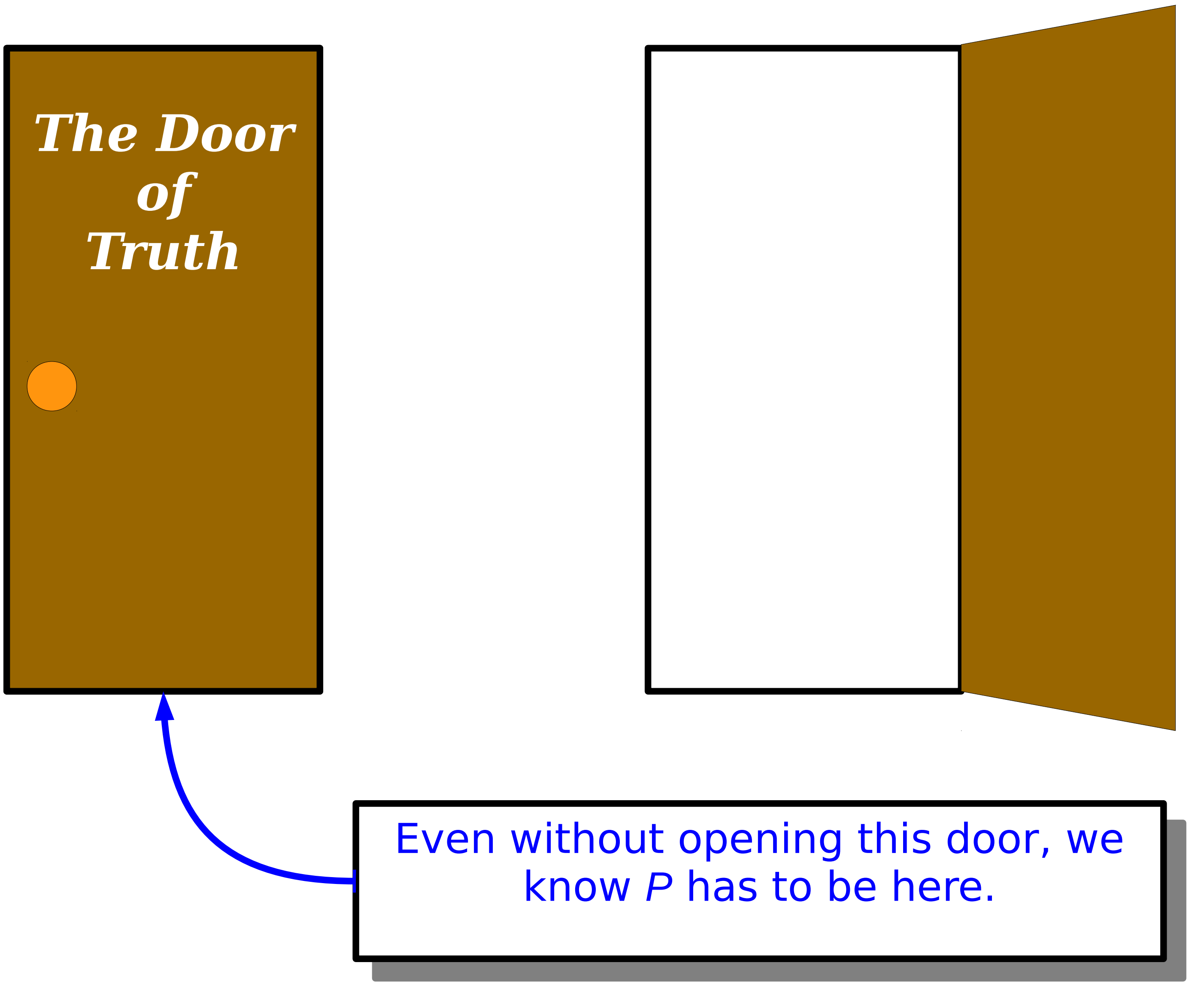

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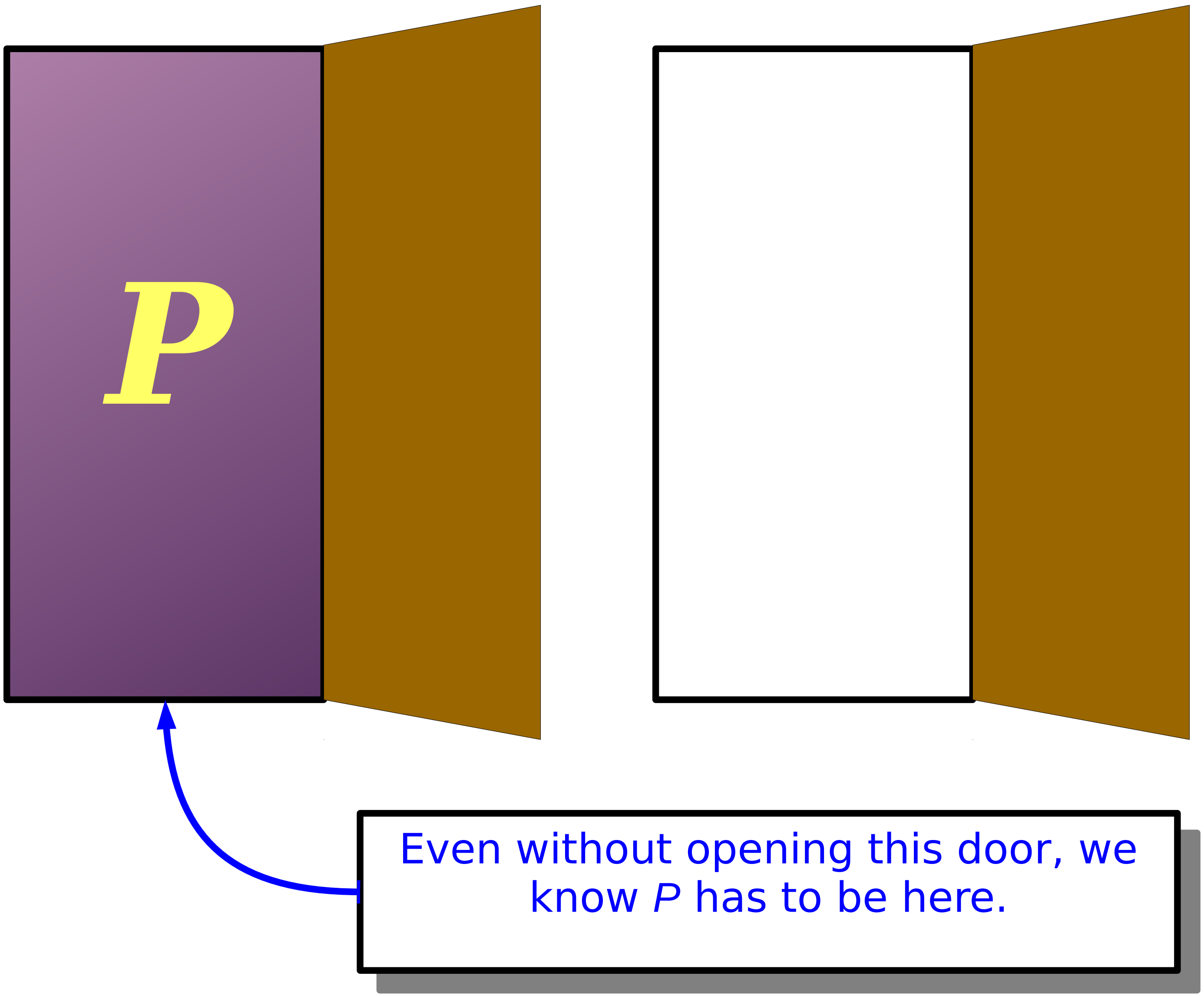

A ***proof by contradiction*** shows that some statement P is true by showing that P isn't false.

Proof by Contradiction

- ***Key Idea:*** Prove a statement P is true by showing that it isn't false.
- First, assume that P is false. The goal is to show that this assumption is silly.
- Next, show this leads to an impossible result.
  - For example, we might have that $1 = 0$, that $x \in S$ and $x \notin S$, that a number is both even and odd, etc.
- Finally, conclude that since P can't be false, we know that P must be true.

An Example: ***Set Cardinalities***

Set Cardinalities

- We've seen sets of many different cardinalities:
  - $|\emptyset| = 0$
  - $|\{1, 2, 3\}| = 3$
  - $|\{ n \in \mathbb{N} \mid n < 137 \}| = 137$
  - $|\mathbb{N}| = \aleph_0$.
- These span from the finite up through the infinite.
- ***Question:*** Is there a "largest" set? That is, is there a set that's bigger than every other set?

Theorem: There is no largest set.

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Proof:

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To prove this statement by contradiction, we're going to assume its negation.

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What is the negation of the statement "there is no largest set?"

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One option: "there *is* a largest set."

Theorem: There is no largest set.

Proof: Assume for the sake of contradiction that there is a largest set; call it S.

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One option: "there *is* a largest set."

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Notice that we're announcing

1. **that this is a proof by contradiction, and**
2. **what, specifically, we're assuming.**

This helps the reader understand where we're going. Remember – proofs are meant to be read by other people!

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The three key pieces:

1. Say that the proof is by contradiction.
2. Say what you are assuming is the negation of the statement to prove.
3. Say you have reached a contradiction and what the contradiction means.

In CS103, please include all these steps in your proofs!

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Proving Implications

- Suppose we want to prove this implication:

  If *P* is true, then *Q* is true.

- We have three options available to us:
  - ***Direct Proof:***
  - ***Proof by Contrapositive.***
  - ***Proof by Contradiction.***

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  - ***Direct Proof:***

    Assume ***P* is true**, then prove ***Q* is true**.
  - ***Proof by Contrapositive.***

    Assume ***Q* is false**, then prove that ***P* is false**.
  - ***Proof by Contradiction.***

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  If *P* is true, then *Q* is true.

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  - ***Direct Proof:***

    Assume ***P* is true**, then prove ***Q* is true**.
  - ***Proof by Contrapositive.***

    Assume ***Q* is false**, then prove that ***P* is false**.
  - ***Proof by Contradiction.***

    … what does this look like?

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What is the negation of our theorem?

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The three key pieces:

1. **Say that the proof is by contradiction.**
2. **Say what the negation of the original statement is.**
3. **Say you have reached a contradiction and what the contradiction entails.**

In CS103, please include all these steps in your proofs!

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Proving Implications

- Suppose we want to prove this implication:

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  - ***Direct Proof:***

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  - ***Proof by Contradiction.***

    Assume ***P is true*** and ***Q is false***, then derive a contradiction.

What We Learned

- ***What's an implication?***
  - It's statement of the form "if P, then Q," and states that if P is true, then Q is true.
- ***How do you negate formulas?***
  - It depends on the formula. There are nice rules for how to negate universal and existential statements and implications.
- ***What is a proof by contrapositive?***
  - It's a proof of an implication that instead proves its contrapositive.
  - (The contrapositive of "if P, then Q" is "if not Q, then not P.")
- ***What's a proof by contradiction?***
  - It's a proof of a statement P that works by showing that P cannot be false.

Your Action Items

- ***Read “Guide to Office Hours,” the “Proofwriting Checklist,” and the “Guide to LaTeX.”***
  - There’s a lot of useful information there. In particular, be sure to read the Proofwriting Checklist, as we’ll be working through this checklist when grading your proofs!
- ***Start working on PS1.***
  - At a bare minimum, read over it to see what’s being asked. That’ll give you time to turn things over in your mind this weekend.

Next Time

- ***Mathematical Logic***
  - How do we formalize the reasoning from our proofs?
- ***Propositional Logic***
  - Reasoning about simple statements.
- ***Propositional Equivalences***
  - Simplifying complex statements.